November 26, 1891.

Mr. JOHN EVANS, D.C.L., LL.D., Treasurer, in the Chair.

A List of the Presents was laid on the table, and thanks ordered for them.

Pursuant to notice, Alexander Agassiz, Dr. Benjamin Apthorpe Gould, Professor Eduard Strasburger, and Professor Pietro Tacchini were balloted for and elected Foreign Members of the Society.

In pursuance of the Statutes, notice of the ensuing Anniversary Meeting was given from the Chair, and the list of Officers and Council nominated for election was read as follows:—

President.—Sir William Thomson, D.C.L., LL.D.

Treasurer.—John Evans, D.C.L., LL.D.

Secretaries.— { Professor Michael Foster, M.A., M.D.
The Lord Rayleigh, M.A., D.C.L.

Foreign Secretary.—Sir Archibald Geikie, LL.D.

Other Members of the Council.—Captain William de Wiveleslie Abney, C.B.; William Thomas Blanford, F.G.S.; Professor Alexander Crum Brown, D.Sc.; Professor George Carey Foster, B.A.; James Whitbread Lee Glaisher, D.Sc.; Frederick Ducane Godman, F.L.S.; John Hopkinson, D.Sc.; Professor George Downing Liveing, M.A.; Professor Joseph Norman Lockyer, F.R.A.S.; Professor Arthur Milnes Marshall, D.Sc.; Philip Henry Pye-Smith, M.D.; William Chandler Roberts-Austen, F.C.S.; Professor Edward Albert Schäfer, M.R.C.S.; Sir George Gabriel Stokes, Bart., M.A.; Professor Sydney Howard Vines, M.A.; General James Thomas Walker, C.B.

The following Papers were read:—

I. "On Instability of Periodic Motion." By Sir WILLIAM THOMSON, P.R.S. Received November 12, 1891.

1. Let $\psi, \phi, \chi, \vartheta$ be generalised coordinates of a system; and let $A(\psi, \phi, \ldots, \psi', \Phi', \ldots)$ be the action in a path (§ 2 above) from the configuration $(\psi', \phi', \ldots)$ to the configuration $(\psi, \phi, \ldots)$ with kinetic energy $(E-V)$ with any given constant value for E, the total energy; V being the potential energy, of which the

value is given for every possible configuration of the system. Let $\nu, \xi, \eta, \zeta, \ldots.$, and $\nu', \xi', \eta', \zeta', \ldots.$, be the generalised component momentums of the system as it passes through the configurations $(\psi, \phi, \ldots.)$ and $(\psi', \phi', \ldots.)$ respectively. If by any means we have fully solved the problem of the motion of the system under the given forcive* (of which V is the potential energy), we know A for every given set of values of $\psi, \phi, \ldots., \psi', \phi', \ldots.$, that is to say, it is a known function of $(\psi, \phi \ldots., \psi', \phi', \ldots.)$. Then, by Hamilton's principle [Thomson and Tait's 'Natural Philosophy,' § 330 (18)], we have

$$\left.\begin{array}{llll} \nu = \dfrac{dA}{d\psi}, & \xi = \dfrac{dA}{d\phi}, & \eta = \dfrac{dA}{d\chi}, & \zeta = \dfrac{dA}{d\vartheta}, \ldots. \\ \nu' = -\dfrac{dA}{d\psi'}, & \xi' = -\dfrac{dA}{d\phi'}, & \eta' = -\dfrac{dA}{d\chi'}, & \zeta' = \dfrac{dA}{d\vartheta'}, \ldots. \end{array}\right\} \ldots (1).$$

2. Now let P′P designate a particular path† from position $(\psi', \phi', \chi', \ldots.)$ which for brevity we shall call P′, to position $(\psi, \phi, \chi, \ldots.)$ which we shall call P. Let ${}_0P'{}_0P$ be a part of a known periodic path, from which P′P is evidently little distant. But first, whether ${}_0P'{}_0P$ is periodic or not, provided it is evidently near to P′P, and provided ${}_0P'$ and ${}_0P$ are infinitely near to P′, and P, respectively, we have, by Taylor's theorem, and by (1),

$$\left.\begin{array}{l} A(\psi, \phi, \chi, \ldots., \psi', \phi, \chi, \ldots.) \\ = A({}_0\psi, {}_0\phi, {}_0\chi, \ldots., {}_0\psi', {}_0\phi', {}_0\chi', \ldots.) \\ \quad + {}_0\nu(\psi - {}_0\psi) + {}_0\xi(\phi - {}_0\phi) + \ldots. - {}_0\nu'(\psi' - {}_0\psi') - {}_0\xi'(\phi' - {}_0\phi') - \ldots. \\ \quad + \frac{1}{2}\left\{ {}_0\!\left(\dfrac{d^2A}{d\psi^2}\right)(\psi - {}_0\psi)^2 + {}_0\!\left(\dfrac{d^2A}{d\phi^2}\right)(\phi - {}_0\phi)^2 + \ldots. \right. \\ \quad \left. + 2\,{}_0\!\left(\dfrac{d^2A}{d\psi\, d\phi}\right)(\psi - {}_0\psi)(\phi - {}_0\phi) + \ldots. \right\} \end{array}\right\} \ldots (2).$$

* This is a term introduced by my brother, Professor James Thomson, to denote a force-system.

† For any given value of E, the sum of potential and kinetic energies, the problem of finding a path from any position P′ to any position P is determinate. Its solution is, for each coordinate of the system, a determinate function of the coordinates which define P and P′ and of t, the time reckoned from the instant of passing through P′. The solution is single for the case of a particle moving under the influence of no force; every path being an infinite straight line. For a single particle moving under the influence of a uniform force in parallel lines (as gravity in small-scale terrestrial ballistics) the solution is duplex or imaginary. For every constrainedly finite system the solution is infinitely multiple; as is virtually well known by every billiard player for the case of a Boscovichian atom flying about within an enclosing surface, and by every tennis player for the parabolas with which he is concerned, and their reflexions from walls or pavement.

3. Let us now simplify by choosing our coordinates so that the values of ϕ, χ, &c., are each zero for every position of the path ${}_0P'{}_0P$; and let ψ, for any position of this path, be the action along it reckoned from zero at ${}_0P'$. These assumptions, expressed in symbols, are as follows:—

$$\left.\begin{array}{l}\dfrac{dA}{d\phi}=0,\quad \dfrac{dA}{d\chi}=0,\ldots \quad \dfrac{dA}{d\psi}=-\psi',\quad \dfrac{dA}{d\phi'}=0,\quad \dfrac{dA}{d\chi'}=0,\ldots \\ \text{for all values of } \psi \text{ and } \psi', \text{ if } \phi=0, \chi=0,\ldots;\ \phi'=0, \chi'=0,\ldots\end{array}\right\} \quad ..(3).$$

4. Taking now

$$\psi=0,\ \psi={}_0\psi,\ {}_0\phi=0,\ {}_0\chi=0,\ldots,\ {}_0\psi'=0,\ {}_0\phi'=0,\ {}_0\chi'=0,\ldots, \quad ..(4);$$

we have

$$A({}_0\psi, {}_0\phi, {}_0\chi, \ldots, {}_0\psi', {}_0\phi', {}_0\chi', \ldots) = A({}_0\psi, 0, 0, \ldots, 0, 0, 0, \ldots) \quad (5)$$

and, in virtue of this and of (3) and (1), (2) becomes

$$\left.\begin{array}{r}A({}_0\psi, \phi, \chi, \ldots, 0, \phi', \chi', \ldots) = A({}_0\psi, 0, 0, \ldots, 0, 0, 0) \\ +\tfrac{1}{2}\,[11\phi^2+22\chi^2+33\vartheta^2+44\phi'^2+55\chi'^2+66\vartheta'^2 \\ +2(12\phi\chi+13\phi\vartheta+14\phi\phi'+15\phi\chi'+16\phi\vartheta' \\ 23\chi\vartheta+24\chi\phi'+25\chi\chi'+26\chi\vartheta' \\ +34\vartheta\phi'+35\vartheta\chi'+36\vartheta\vartheta' \\ +45\phi'\chi'+46\phi'\vartheta' \\ +56\chi'\vartheta')]\end{array}\right\} \quad ..(6),$$

where, merely for simplicity of notation, we suppose the total number of freedoms of the system, that is to say, the total number of the coordinates ψ, ϕ, χ, ϑ, to be four; and for brevity put

$$ {}_0\!\left(\frac{d^2A}{d\phi^2}\right)=11,\quad {}_0\!\left(\frac{d^2A}{d\phi\, d\chi}\right)=12,\quad {}_0\!\left(\frac{d^2A}{d\chi^2}\right)=22, \text{ \&c.} \quad \ldots\ldots (7).$$

5. From (6) we find, by (1),

$$\left.\begin{array}{r}\xi=11\phi+12\chi+13\vartheta+14\phi'+15\chi'+16\vartheta' \\ \eta=21\phi+22\chi+23\vartheta+24\phi'+25\chi'+26\vartheta' \\ \zeta=31\phi+32\chi+33\vartheta+34\phi'+35\chi'+36\vartheta' \\ -\xi'=41\phi+42\chi+43\vartheta+44\phi'+45\chi'+46\vartheta' \\ -\eta'=51\phi+52\chi+53\vartheta+54\phi'+55\chi'+56\vartheta' \\ -\zeta'=61\phi+62\chi+63\vartheta+64\phi'+65\chi'+66\vartheta'\end{array}\right\} \ldots\ldots\ldots (8).$$

These equations allow us to determine the three displacements, ϕ, χ, ϑ, and the three corresponding momentums, ξ, η, ζ, for any position on the path, in terms of the initial values, ϕ', χ', ϑ', ξ', η', ζ', supposed known.

6. To introduce now our supposition (§ 2) that ${}_0\mathrm{P}'{}_0\mathrm{P}$ is part of a periodic path; let Q be a position on it between ${}_0\mathrm{P}'$ and ${}_0\mathrm{P}$; and let us now, to avoid ambiguity, call it ${}_0\mathrm{P}'\mathrm{Q}_0\mathrm{P}$. Let ${}_0\mathrm{P}'$ and ${}_0\mathrm{P}$ now be taken to coincide in a position which we shall call O; in other words, let ${}_0\mathrm{P}'\mathrm{Q}_0\mathrm{P}$, or OQO, be the complete periodic circuit, or orbit as we may call it. Our path P′P is now a path infinitely near to this orbit, and P′ and P are two consecutive positions in it for which ψ has the value zero. These two positions are infinitely near to one another and to O. We shall call them O_i, and O_{i+1}, considering them as the positions on our path in which ψ is zero for the ith time and for the $(i+1)$th time, from an earlier initial epoch than first passage through $\psi = 0$ which we have been hitherto considering. It is accordingly convenient now to modify our notation as follows:—

$$\left.\begin{array}{llllll} \phi' = \phi_i, & \chi' = \chi_i, & \vartheta' = \vartheta_i; & \xi' = \xi_i, & \eta' = \eta_i, & \zeta' = \zeta_i \\ \phi = \phi_{i+1}, & \chi = \chi_{i+1}, & \vartheta = \vartheta_{i+1}; & \xi = \xi_{i+1}, & \eta = \eta_{i+1}, & \zeta = \zeta_{i+1} \end{array}\right\} \ldots (9).$$

Here ϕ_i, χ_i, ϑ_i are the generalised components of distance from O, at the ith transit through $\psi = 0$ of the system pursuing its path infinitely near to the orbit; and ξ_i, η_i, ζ_i are the corresponding momentum components. With the notation of (9), equations (8) become equations by which the values of these components for the $i+1$th time of transit through $\psi = 0$ can be found from their values for the ith time. They are equations of finite differences, and are to be treated *secundum artem*, as follows:—

7. Assume

$$\left.\begin{array}{lll} \phi_{i+1} = \rho\phi_i, & \chi_{i+1} = \rho\chi_i, & \vartheta_{i+1} = \rho\vartheta_i; \\ \xi_{i+1} = \rho\xi_i, & \eta_{i+1} = \rho\eta_i, & \zeta_{i+1} = \rho\zeta_i \end{array}\right\} \ldots\ldots\ldots (10).$$

Substituting accordingly in (8) modified by (9), and eliminating ξ_i, η_i, ζ_i, we find

$$\left.\begin{array}{l} \left(11+\dfrac{14}{\rho}+41\rho+44\right)\phi+\left(12+\dfrac{15}{\rho}+42\rho+45\right)\chi+\left(13+\dfrac{16}{\rho}+43\rho+46\right)\vartheta = 0 \\ \left(21+\dfrac{24}{\rho}+51\rho+54\right)\phi+\left(22+\dfrac{25}{\rho}+52\rho+55\right)\chi+\left(23+\dfrac{26}{\rho}+53\rho+56\right)\vartheta = 0 \\ \left(31+\dfrac{34}{\rho}+61\rho+64\right)\phi+\left(32+\dfrac{35}{\rho}+62\rho+65\right)\chi+\left(33+\dfrac{36}{\rho}+63\rho+66\right)\vartheta = 0 \end{array}\right\} (11).$$

Remarking that $41 = 14$, $12 = 21$, &c., we see that the determinant

for the elimination of the ratios $\phi|\chi|\vartheta$ is symmetrical with reference to ρ and $1|\rho$. Hence it is

$$C_3(\rho^3+\rho^{-3})+C_2(\rho^2+\rho^{-2})+C_1(\rho+\rho^{-1})+2C_0 \ldots\ldots\ldots (12),$$

where C_0, C_1, C_2, C_3 are coefficients of which the values in terms of 11, 12, &c., are easily written out. The determinant equated to zero gives an equation of the sixth degree for determining ρ, of which for each root there is another equal to its reciprocal. We reduce it to an equation of the third degree by putting

$$\rho+\rho^{-1}=2e \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

Let e_1, e_2, e_3 be the roots of the equation thus found. The corresponding values of ρ are

$$e_1 \pm \sqrt{(e_1^2-1)}\,;\ \ e_2 \pm \sqrt{(e_2^2-1)}\,;\ \ e_3 \pm \sqrt{(e_3^2-1)} \ldots\ldots (14).$$

In the case of e having any real value between 1 and -1, it is convenient to put

$$\left.\begin{array}{lr} & e=\cos\alpha, \\ \text{which gives} & \rho=\cos\alpha+\iota\sin\alpha \\ \text{and} & \rho^{-1}=\cos\alpha-\iota\sin\alpha \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (15).$$

8. Suppose now, for the first time of passing through $\psi=0$, the three coordinates and three corresponding momentums, ϕ_1, χ_1, ϑ_1, ξ_1, η_1, ζ_1, to be all given; we find

$$\left.\begin{array}{l} \phi_{i+1}=A_1\rho_1^{\,i}+A'_1\rho_1^{-i}+A_2\rho_2^{\,i}+A'_2\rho_2^{-i}+A_3\rho_3^{\,i}+A'_3\rho_3^{-i} \\ \chi_{i+1}=B_1\rho_1^{\,i}+B'_1\rho_1^{-i}+B_2\rho_2^{\,i}+B'_2\rho_2^{-i}+B_3\rho_3^{\,i}+B'_3\rho_3^{-i} \\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \xi_{i+1}=F_1\rho_1^{\,i}+F'_1\rho_1^{-i}+F_2\rho_2^{\,i}+F'_2\rho_2^{-i}+F_3\rho_3^{\,i}+F'_3\rho_3^{-i} \end{array}\right\} \ldots (16),$$

where A_1, A'_1, A_2, A'_2, ..., F_1, F'_1, F_2, F'_2 are thirty-six coefficients which are determined by the six equations (16), with $i=0$: and the six equations (8), modified by (9); with i successively put $=1$, 2, 3, 4, 5; with the given values substituted for ϕ_1, χ_1, ϑ_1, ξ_1, η_1, ζ_1 in them; and with for ϕ_2, χ_2, &c., their values by (16).

9. Our result proves that every path infinitely near to the orbit is unstable unless every root of the equation for e has a real value between 1 and -1. It does not prove that the motion *is* stable when this condition is fulfilled. Stability or instability for this case cannot be tested without going to higher orders of approximation in the consideration of paths very nearly coincident with an orbit.

ADDENDUM.

The subject of periodic motion and its stability has been treated with great power by M. Poincaré in a paper, " Sur le Problème des Trois Corps et les Équations de la Dynamique," for which the prize of His Majesty the King of Sweden was awarded on the 21st January, 1889. This paper, which has been published in Mittag-Leffler's 'Acta Mathematica,' 13, 1 and 2 (270 4to pp.), Stockholm, 1890, only became known to me recently through Professor Cayley. I am greatly interested to find in it much that bears upon the subject of my communication of last June to the Royal Society " On some Test Cases for the Maxwell-Boltzmann Doctrine regarding Distribution of Energy;" particularly in p. 239, the following paragraph :— " On peut démontrer que dans le voisinage d'une trajectoire fermée représentant une solution périodique, soit stable, soit instable, il passe une infinité d'autres trajectoires fermées. Cela ne suffit pas, en toute rigueur, pour conclure que toute région de l'espace, si petite qu'elle soit, est traversée par une infinité des trajectoires fermées, mais cela suffit pour donner à cette hypothèse un haut caractère de vraisemblance."* This statement is exceedingly interesting in connexion with Maxwell's fundamental supposition quoted in § 10 of my paper, " that the system, if left to itself in its actual state of motion, will, sooner or later, pass through every phase which is consistent with the equation of energy ; "† an assumption which Maxwell gives not as a conclusion, but as a proposition which " we may with considerable confidence assert, ... except for particular forms of the surface of the fixed obstacle." It will be seen that Poincaré's " hypothesis, having a high character of probability," does not go so far as Maxwell's, which asserts that every portion of space is traversed *in all directions* by *every trajectory*. The conclusion which I gave in § 13, as seeming to me quite certain, " that every mode differs infinitely little from being a fundamental mode," is clearly a necessary consequence of Maxwell's fundamental supposition; the truth of which still seems to me highly probable, *provided exceptional cases are properly dealt with.*

I also find the following statement, pp. 100—101 :—" Il y aura donc en général n quantités α^2 distinctes. Nous les appellerons les *coefficients de stabilité* de la solution périodique considérée.

" Si ces n coefficients sont tous réels et négatifs, la solution périodique sera stable, car les quantités ξ_i and η_i, resteront inférieures à une limite donnée.

" Il ne faut pas toutefois entendre ce mot de stabilité au sens

* The "trajectoire fermée" of M. Poincaré is what I called a "fundamental mode of rigorously periodic motion," or "an orbit."

† 'Scientific Papers,' vol. 2, p. 714.

absolu. En effet, nous avons négligé les carrés des ξ et des η et rien ne prouve qu'en tenant compte de ces carrés, le résultat ne serait pas changé. Mais nous pouvons dire au moins que les ξ et η, s'ils sont originairement très petits, resteront très petits pendant très longtemps. Nous pouvons exprimer ce fait en disant que la solution périodique jouit sinon de la stabilité *séculaire*, du moins de la stabilité *temporaire*." Here the conclusion of § 9 of my present paper is perfectly anticipated, and is expressed in a most interesting manner. M. Poincaré's investigation and mine are as different as two investigations of the same subject could well be, and it is very satisfactory to find perfect agreement in conclusions.

II. "A new Mode of Respiration in the Myriapoda." By F. G. Sinclair (formerly F. G. Heathcote), M.A., Fellow of the Cambridge Philosophical Society. Received August 12, 1891.

(Abstract.)

The Scutigeridæ respire by means of a series of organs arranged in the middle dorsal line at the posterior edge of every dorsal scale except the last.

Each organ consists of a slit bounded by four curved ridges, two at the edges of the slit, and two external to the latter. The slit leads into an air sac. From the sac a number of tubes are given off; these tubes are arranged in two semicircular masses. The ends of the tubes project into the pericardium in such a manner that the ends are bathed in the blood and aërate it just before it is returned into the heart by means of the ostia. In the living animal the blood can be seen through the transparent chitin of the dorsal surface surrounding the ends of the tubes; and in the organ and surrounding tissues cut out of a Scutigera directly it is killed, the blood corpuscles can be seen clustering round the tube ends. If the mass of tubes of a freshly killed specimen are teased out under the microscope in glycerine, they can be seen to be filled with air. The tubes each branch several times. Each tube is lined with chitin, which is a continuation of the chitin of the exo-skeleton. Each tube is also clothed with cells, which are a continuation of the hypodermis. The tubes end in a blunt point of very delicate chitin.

Reasons for supposing these Organs to be Respiratory.

1. There are no other organs which could be supposed to be respiratory in function.

2. The tubes are chitinous, and the chitin grows thin and mem-